Answer all the questions.

The figure in the right side

indicate marks

Part-I

1. **Answer ALL the questions.**   **(1x10=10)**

   (i)   Define countable set.

   (ii)   If $S = \left\{ \frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{N} \right\}$, then find $Sup\ S\ and\ Inf\ S$.

   (iii)   Give an example satisfying nested interval property.

   (iv)   The sequence $(x_n) = \frac{1}{2^n}$ is a Cauchy Sequence.(True/False)

   (v)   Give an example of a sequence which is not convergent, but two of its subsequences are convergent.

   (vi)   Define absolute convergence of a series.

   (vii)   The series $\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n} \right)$ diverges.(True/False)

   (viii)   The series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges. (True/False)

   (ix)   What is Lipschitz condition for a function ?

   (x)   State location of root theorem .

Part-II

2. **Answer ALL questions**   **(2x9=18)**

   (i)   Solve: $|4x - 5| \le 13$.

   (ii)   Use squeeze theorem to show that $\lim_{n \to \infty} \frac{\sin n}{n} = 0$.

   (iii)   Show that the set of integers $\mathbb{Z}$ is countable.

   (iv)   Show that $\lim_{n \to \infty} \frac{1}{n^2+1} = 0$.

   (v)   Show that $\lim_{n \to \infty} n^{1/n} = 1$.

   (vi)   Show that if a series $\sum x_n$ is convergent , then $\lim_{n \to \infty} x_n = 0$.

   (vii)   Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{\log n}$

   (viii)   Test the convergence of *the series* $\sum_{n=1}^{\infty} \frac{1}{n!}$

   (ix)   Show that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

Part-III

3. **Answer any Eight questions from following.**   **(5x8=40)**

   (i)   Prove the rational density theorem.

   (ii)   Show that the set $\mathbb{N} \times \mathbb{N}$ is denumerable.

   (iii)   Show that the sequence $(x_n) = \left(1 + \frac{1}{n}\right)^n$ is bounded and converges to the irrational number e.

   (iv)   Prove that a Cauchy sequence of real numbers is bounded.

   (v)   Test the convergence of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots$

   (vi)   Test the convergence of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$

   (vii)   Prove that if a series in $\mathbb{R}$ is absolutely converge, then it is convergent. Is the converse of this statement is true . Justify your answer.

(viii)   Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$, $f(x) \geq 0$, for all $x \in A$ and $(\sqrt{f})(x) = \sqrt{f(x)}$. Then show that if f is continuous at $c \in A$, then $(\sqrt{f})$ is continuous at c.

(ix)   If $f(x) = x^2$ and $A = [0, b]$, $b > 0$, then show that f is uniformly continuous on A.

(x)   Show that $\lim\limits_{x \to 0} \dfrac{1}{x}$ does not exits in $\mathbb{R}$.

Part-IV

4.   **Answer any Four question from following.**   **(8x4=32)**

(i)   Show that union of family of countable sets is countable i.e If A_1, A_2, A_3 ... ... be a sequence of countable sets, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

(ii)   State and prove Bolzano-Weierstrass theorem for sequence.

(iii)   Let $X = (x_n)$ be a sequence defined by $x_1 = 1$, $x_2 = 2$ and $x_n = \dfrac{1}{2}(x_{n-2} + x_{n-1})$, for $n > 2$. Show that $X = (x_n)$ is a Cauchy sequence.

(iv)   State and prove Maximum-Minimum theorem.

(v)   Let I be a closed and bounded interval and $f: I \to \mathbb{R}$ be continuous on I. Then show that f is uniformly continuous on I.

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April, 2025
Algebra – I(Major)

Full Marks – 100   Time – 3 hrs

Figures in the right-hand margin indicates marks
Part – I

1. Answer all questions.   **[1x10 = 10 marks]**

a) $(1235)(234)(1523)$ in S_5 is an even permutation.(True/False)

b) Is $\phi : (\mathbb{R}, +) \to (\mathbb{R}^+, \times)$ defined by $\phi(x) = 2^x$ a group isomorphism? (Yes/No)

c) If H is a subgroup of G with $|G| = 12$ and $|H| = 6$. How many cosets does H have?

d) Define a normal subgroup H of a group G.

e) $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ defined as $\phi(x) = x^n$ is group homomorphism iff $n = $ _____.

f) What is the characteristic of $\mathbb{Q}$?

g) Define an ideal I of a commutative ring R.

h) What is the unity in the ring $R = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} : a \in \mathbb{R} \right\}$?

i) Is $x^2 + 1$ irreducible over $\mathbb{Z}_2$? (Yes/No)

j) Is every Euclidean domain a unique factorization domain? (Yes/No)

Part – II

2. Answer all questions.   **[2x9 = 18 marks]**

a) Express the permutation $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{bmatrix}$ in cycle notation in S_6.

b) Find the elements of factor group $\dfrac{(\mathbb{Z}_3 \times \mathbb{Z}_2)}{(\{0\} \times \mathbb{Z}_2)}$.

c) Show that a group of order prime is always cyclic.

d) Show that kernel of a group homomorphism is normal.

e) Is a prime ideal always maximal? Justify.

f) Find $(3x^3 - 2x^2 + x - 1)(2x^2 + x - 1)$ in $\mathbb{Z}_4[x]$.

g) Let F be a field and $f(x) \in F[x]$. Show that a is a zero of $f(x)$ iff $(x - a) | f(x)$.

h) Is $5x^3 - 27x^2 + 96x - 21$ irreducible over $\mathbb{Q}$? Justify.

i) Give example of a Euclidean domain and a unique factorization domain.

Part – III

3. Answer any eight questions.   **[5x8 = 40 marks]**

a) Define the alternating subgroup A_n of degree n, and show that $|A_n| = \dfrac{n!}{2}$.

b) Let the groups G and G' be isomorphic. Show that G is abelian iff G' is abelian.

c) Let H be a subgroup of G and $a \in G$. Show that either $aH = bH$ or $aH \cap bH = \emptyset$.

d) Show that a group of order 75 can have at most one subgroup of order 25.

e) Define center of a group and show that it is a normal subgroup.

f) Prove that characteristic of an integral domain is either 0 or a prime.

g) If $\phi : R \to S$ is ring homomorphism. Show that ϕ is one-one iff ker $\phi = \{0\}$.

h) Let F be a field and $f(x) \in F[x]$ with $2 \leq \deg f(x) \leq 3$. Show that $f(x)$ is reducible over F iff $f(x)$ has a zero in F.

i) Show that $1 + \sqrt{-3}$ is an irreducible element in the integral domain $\mathbb{Z}[\sqrt{-3}]$.

j) For a prime p, prove that $\Phi_p(x) = \frac{x^p - 1}{x - 1}$ is irreducible over $\mathbb{Q}$.

Part – IV

4. Answer any four questions.   **[8x4 = 32 marks]**

a) Show that every group is isomorphic to a group of permutations.

b) Let $\phi : G \to G'$ be a group homomorphism. Show that $G / \ker \phi \approx \phi(G)$.

c) Let R be a ring and $A \subseteq R$ be a subring. Show $R/A = \{r + A \mid r \in R\}$ is a ring under the operations $(s + A) + (t + A) = (s + t) + A$ and $(s + A)(t + A) = st + A$ iff A is an ideal of R.

d) Let F be a field and $f(x), g(x) \neq 0 \in F[x]$. Then there exist unique $q(x), r(x) \in F[x]$ such that $f(x) = g(x)q(x) + r(x)$ with either $r(x) = 0$ or $\deg r(x) < \deg g(x)$. Prove it.

e) Show that in a principal ideal domain an element is irreducible iff it is a prime.

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April,2025
Calculus and Analytic Geometry(Minor)

Full Marks: 100   Time: 3 hours

Figures in the right-hand margin indicates marks. Answer as per directions

1. Answer all the questions   **(1 x 10=10)**
    (i)   The Equation $sinhx = coshx$ has _____number of Solution.

         (a)   Zero   (b)   One

         (c)   Two   (d)   Infinite

    (ii)   The inflection point of $f(x) = x^4$ is zero. (True/False)

    (iii) $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x =$ _____

    (iv) $\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) =$ _____

    (v) $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx =$ _____.

    (vi)   Classify the conic $x^2 - xy + y^2 - 2 = 0$

         (a)   Hyperbolic   (b)   Parabolic

         (c)   Elliptic   (d)   Circle

    (vii)   Write the parametric equation of the parabola $y^2 = 4ax$

    (viii)   The value of λ so that the vectors $\vec{a} = \hat{k}, \vec{b} = \lambda\hat{\imath} - 3\hat{k}, \vec{c} = \hat{\jmath}$ are Coplanar is 0.
             (True/False)

    (ix)   If $\vec{a}$ and $\vec{b}$ are orthogonal unit vector then find the value of $(\vec{a} \, X\vec{b})X\vec{a}$ is ____.

    (x)   The arc length of the curve $y = x$ on the interval [0,4] is_____.

2. Answer all the questions.(Very short answer type question Max. 50 Words)   **(2 x 9=18)**
    (i)   Find the horizontal asymptotes of the curve $a^2(x^2 + y^2) = x^2y^2$

    (ii)   Find the value of $\lim_{x \to 0}(\frac{1}{x} - \frac{1}{sinx})$

    (iii)   Find $\int \cosh^3 x . sinhx \, dx$

    (iv)   Express the integral $\int_0^1 \frac{x}{x+1} \, dx$ as limit of Riemann sums (do not evaluate the integral).

    (v)   Evaluate $\int \frac{logx}{x} dx$

    (vi)   Find the volume of the solid generated when $y = x^3, x = 1, y = 0$ is revolved about y axis.

    (vii)   Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1, x = 2$ $about$ y $axis$.

    (viii)   Find the vertex and directrix of the parabola $y^2 = -8x$.

    (ix)   If $\vec{a} = \hat{\imath} - 5\hat{k}, \vec{b} = 2\hat{\imath} - 4\hat{\jmath} + \hat{k}, \vec{c} = 3\hat{\imath} - 2\hat{\jmath} + 5\hat{k}$ $find$ $\vec{a}.(\vec{b} \, X\vec{c})$

3.   Answer any **Eight** questions   **(5 x 8=40)**

(i)   Find the interval in which the curve $y = (\cos x + \sin x)e^x$ is concave up and concave
        down ,for $x \in (0,2\pi)$.

(ii)   If $x = \theta - \sin \theta$, $y = 1 - \cos \theta$ then find $\frac{d^2 y}{dx^2}$ at $\theta = \pi$.

(iii)   Find the asymptotes of the curve $6x^3 - 11x^2 y + 6xy^2 - y^3 + 2x + 3y + 7 = 0$.

(iv)   Evaluate $\int \ln(1 + x^2)dx$.

(v)   Evaluate $\int \sin^4 x \cos^3 x \, dx$.

(vi)   State and prove Reflection property of Parabola.

(vii)   Find the arc length of the curve $x = t - \sin t, y = 1 - \cos t, for \ 0 \le t \le 2\pi$.

(viii) Find the volume of the solid region enclosed by the curve $y = x^2, y = x^3$ when revolved
        about $x \ axis$.

(ix)   Find the tangential and normal component of the acceleration of an object that moves with
        the position vector $r(t) = \sin t \, \hat{\imath} + \cos t \, \hat{\jmath} + \sin t \, \hat{k}$

(x)   Find $\lim\limits_{t \to 1}(\frac{t^4 - 1}{t - 1}\hat{\imath} + \frac{t^2 - 3t + 2}{t^2 + t - 2}\hat{\jmath} + \frac{1 - \cos t}{t} \, \hat{k})$

4.   Answer any **Four** questions.   **(8 x 4=32)**

(i)   If $y = (\sin^{-1} x)^2$ then show that
        $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2 y_n = 0$. also find $y_n(0)$.

(ii)   Trace the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

(iii)   Derive the formula for $\int \sec^n x \, dx$ and use it to find $\int \sec^4 x \, dx$.

(iv)   Find the equation of the ellipse with foci
        $(2, -2), (4, -2)$ and vertices at $(1, -2), (5, -2)$.

(v)   If $\vec{a} = t\hat{\imath} + 3\hat{\jmath} + 2t\hat{k}, \vec{b} = \hat{\imath} - 2\hat{\jmath} + 2\hat{k}, \vec{c} = 3\hat{\imath} + t\hat{\jmath} - \hat{k}. find \ \int_1^2 (\vec{a} \, X(\vec{b} \, X\vec{c})dt$

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